PMM U.S.S.R,Vol.45.pp.598-603 0021-8928/82/5 0598 $7.50/0

UDC 531.38

THE GENERAL CASE OF LIAPUNOV'S PERIODIC MOTIONS OF A HEAVY SOLID BODY WITH A SINGLE FIXED POINT*

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Geometric interpretation is given for a class of periodic motions of a dynamically unsymmetric heavy solid body, which exist according to Liapunov's theorem on the holomorphic integral close to permanent rotations. Orientation and the ratio of semiaxes of ellipses of small oscillations (first approximation trajectories) with respect to the Staude curve of permanent rotations are investigated.

The case when the center of mass of a solid body lies on the principal axis of its ellipsoid of inertia is analytically considered besides a number of singularities of the general case. Investigation of the latter is supplemented by results obtained on a computer for the characteristic example which is of considerable interest because of the complexity constructing the Staude curve. The analysis enabled us to elucidate the properties of Liapunov's periodic motions and their first approximation which differ from the previously obtained /1,2/.

1. The motion of a solid body with a fixed point (suspended solid particle) under the action of gravity is defined by the Euler—Poisson equations which, after linearization near permanent rotation (relative to the equilibrium position), are of the form

$$I_1\xi_1^{\cdot} = (I_2 - I_3)(\omega_2\xi_3 + \omega_3\xi_2) + \nu_2 g_3 - \nu_3 g_2 \qquad (1.1)$$
$$\nu_1^{\cdot} = \omega_3\nu_2 - \omega_2\nu_3 + \xi_3\gamma_2 - \xi_2\gamma_3 \qquad (1\ 2\ 3)$$

where g_i are coordinates of the body center of mass multiplied by its weight, I_i are moments of inertia, ω_i, γ_i are components of vectors of angular velocity $\boldsymbol{\omega}$ and of the vertical $\boldsymbol{\gamma}$ (which defines permanent rotation), respectively in the system of coordinates attached to the principal axes of the ellipsoid of inertia with ξ_i, ν_i their respective variations, $i = 1, 2, 3$, and the symbol (123) indicating omitted equations (or terms of a sum) that are obtained by cyclic permutation of indices.

Complete equations of motion imply that ω_i and γ_i are connected by the relations

$$\gamma_1 = \omega_1/\omega, \ (I_2 - I_3)\,\omega^2\gamma_2\gamma_3 = \gamma_3 g_2 - \gamma_2 g_3 \qquad (1\ 2\ 3) \qquad (1.2)$$

so that the permanent rotations fill the Staude curve (K), i.e. that part of the intersection of cone

$$\sum_{(123)} (I_2 - I_3)\, g_1\gamma_2\gamma_3 = 0 \qquad (1.3)$$

with the unit sphere

$$\sum_{(123)} \gamma_1^2 = 1 \qquad (1.4)$$

where $\omega^2 \geqslant 0$.

In the permanent rotation neighborhood the small oscillation trajectories

$$\mathbf{v} = c\,(\mathbf{u}\cos\Omega t + \mathbf{v}\sin\Omega t) \qquad (1.5)$$

are ellipses with principal directions

$$\mathbf{u} = \mathbf{u}'\cos\varphi + \mathbf{v}'\sin\varphi, \ \mathbf{v} = -\mathbf{u}'\sin\varphi + \mathbf{v}'\cos\varphi$$

provided that angle φ has been chosen so that the scalar product $(\mathbf{u}, \mathbf{v}) = 0$. Here $\mathbf{u}' + i\mathbf{v}'$ is the coordinate component of the eigenvector of system (1.1) with pure imaginary eigenvalue $\lambda = -i\Omega$.

*Prikl.Matem.Mekhan.,45,No.5,800-807,1981

When point (1.2) moves on K, two quantities that completely define the shape and orientation of ellipses lying in the plane tangent to sphere (1.4) change at point (1.2). These quantities are the ratio of semiaxes $f = v/u$ and the angle $\theta = \arccos[(u, z)/(uz)]$ between vector $\mathbf{u}$ and the normal $\mathbf{z}$ to cone (1.3) with components

$$z_1 = (I_1 - I_2)\, g_3\gamma_2 + (I_3 - I_1)\, g_2\gamma_3 \qquad (1\ 2\ 3)$$

For the determination of θ and f (as functions of a point on K) it is, thus, necessary and sufficient to know at every point the pair $\mathbf{u}, \mathbf{v}$.

Statement 1. Vectors $\mathbf{u}$ and $\mathbf{v}$ are of the form

$$\mathbf{u} = [\mathbf{R}\cos\varphi + \mathbf{S}\sin\varphi,\ \gamma],\quad \mathbf{v} = [-\mathbf{R}\sin\varphi + \mathbf{S}\cos\varphi,\ \gamma] \tag{1.6}$$

$$\operatorname{tg} 2\varphi = 2\frac{(R, S) - (R, \gamma)(S, \gamma)}{R^2 - S^2 - (R, \gamma)^2 + (S, \gamma)^2}$$

$$\mathbf{R} = (R_1, R_2, R_3),\quad \mathbf{S} = (S_1, S_2, S_3)$$

$$R_1 = -\frac{g_2\gamma_2}{I_1 - I_2}(\alpha_1\alpha_2 - 2\gamma_3^2) + \frac{g_3\gamma_3}{I_3 - I_1}(2\gamma_2^2 - \alpha_1\alpha_3) - \lambda\omega\Big(\alpha_1\alpha_2\alpha_3 - \sum_{(1\,2\,3)} \alpha_1\gamma_1^2\Big)$$

$$R_2 = \frac{g_1\gamma_2}{I_1 - I_2}(\alpha_1\alpha_2 - 2\gamma_3^2) + 2\omega^2\gamma_1\gamma_2\gamma_3^2$$

$$R_3 = \frac{g_1\gamma_3}{I_3 - I_1}(\alpha_1\alpha_3 - 2\gamma_2^2) - 2\omega^2\gamma_1\gamma_2^2\gamma_3$$

$$iS_1 = -\frac{g_2\alpha_1\gamma_1\gamma_3}{I_1 - I_2} - \frac{g_3\alpha_1\gamma_1\gamma_2}{I_3 - I_1} - 2\lambda\omega\gamma_1\gamma_2\gamma_3$$

$$iS_2 = \frac{g_1\alpha_1\gamma_1\gamma_3}{I_1 - I_2} - \frac{g_3}{I_2 - I_3}(\alpha_3\gamma_3^2 - \alpha_2\gamma_2^2) + \omega^2\gamma_3\Big(\alpha_1\alpha_2\alpha_3 - \sum_{(1\,2\,3)} \alpha_1\gamma_1^2\Big)$$

$$iS_3 = \frac{g_1\alpha_1\gamma_1\gamma_2}{I_3 - I_1} + \frac{g_2}{I_2 - I_3}(\alpha_3\gamma_3^2 - \alpha_2\gamma_2^2) - \omega^2\gamma_2\Big(\alpha_1\alpha_2\alpha_3 - \sum_{(1\,2\,3)} \alpha_1\gamma_1^2\Big)$$

$$\alpha_1 = -\frac{\lambda I_1}{\omega(I_2 - I_3)} \quad (1\ 2\ 3)$$

Proof. Expressing ξ_i in terms of ν_i in the first three equations of system (1.1), for the determination of eigenvectors we have

$$\xi_1 = \delta^{-1}[\zeta_1(\alpha_2\alpha_3 - \gamma_1^2) + \zeta_2(\gamma_1\gamma_2 - \alpha_3\gamma_3) + \zeta_3(\gamma_1\gamma_3 - \alpha_2\gamma_2)] \qquad (1\ 2\ 3)$$

$$\zeta_1 = (\nu_3 g_2 - \nu_2 g_3)\ [(I_2 - I_3)\,\omega]^{-1} \qquad (1\ 2\ 3)$$

$$\delta = 2\gamma_1\gamma_2\gamma_3 + \alpha_1\alpha_2\alpha_3 - \sum_{(1\,2\,3)} \alpha_1\gamma_1^2$$

From the fourth equation of the system we obtain $(\mathbf{u}' + i\mathbf{v}',\ \mathbf{R} + i\mathbf{S}) = 0$, where $\mathbf{R}$ and $\mathbf{S}$ are defined by formulas (1.6), and from the last three equations we have $(\mathbf{u}' + i\mathbf{v}',\ \gamma) = 0$. It is consequently possible to assume $\mathbf{u}' + i\mathbf{v}' = [\mathbf{R} + i\mathbf{S},\ \gamma]$. Finally, angle φ in formula (1.6) is determined by formula

$$\operatorname{tg} 2\varphi = 2\,(\mathbf{u}', \mathbf{v}')\ (u'^2 - v'^2)^{-1} \tag{1.7}$$

which ensures the fulfillment of condition $(\mathbf{u}, \mathbf{v}) = 0$.

Remark. Since angle φ is determined with an accuracy to $\pi/2$, the directions of $\mathbf{u}, \mathbf{v}$ are accurate within the substitution $(\pm)\,\mathbf{v}, (\mp)\,\mathbf{u}$.

2. If the parameters of the body are free of constraints, the dependence of functions θ and f on the point position of curve K is complex (see Sects.3 and 4 below). However the picture is sometimes simplified.

Statement 2. If the body center of mass is on the principal axis, vectors $\mathbf{u}$ and $\mathbf{v}$ are directed along the normal and the tangent to the Staude curve, respectively, i.e. $\theta \equiv 0$.

In fact, curve K decomposes into two coordinate semicircles. The motion trajectories prove to be symmetric relative to the planes of these semicircles $\mathbf{u} \parallel \mathbf{z}$, $\mathbf{v} \parallel \mathbf{z}$, hence $\theta \equiv 0$, which is also implied by the general formulas (1.6). Statement 2 also holds when the body center of mass is in the equatorial plane (for admissible points on the coordinate semicircle), as well as in the case of dynamic symmetry investigated in /4/.

Thus, when the body center of mass lies on the principal axis, it is necessary to analyze the behavior of function f. Let for definiteness $g_2 = g_3 = 0 < g_1$, $I_1 = mI_3$, $I_2 = nI_3$. Permanent rotations fill semicircles $\gamma_2 = 0, \gamma_1 > 0$ when $m > 1$ ($\gamma_2 = 0, \gamma_1 < 0$, when $m < 1$), $\gamma_3 = 0, \gamma_1 > 0$ when $m > n$ ($\gamma_3 = 0, \gamma_1 < 0$, when $m < n$), and the isolated points $\gamma_1 = \pm 1$. Let us restrict the investigation to the case of $\gamma_2 = 0$, assuming that no constraints are imposed on m and n except the triangle inequality

$$m + n \geqslant 1, \quad n - 1 \leqslant m \leqslant n + 1 \tag{2.1}$$

From formulas for **u** and **v** follow two equivalent expressions for f

$$f = \sqrt{l(1-m)}\left|\frac{lmn + (n-1)\gamma_3^2 + m(m-n-1)\gamma_1^2}{lmn + (n-1)\gamma_3^2 - m(n-1)\gamma_1^2}\right| = \sqrt{l(1-m)}\left|\frac{m + (n-1)\gamma_3^2 - m(m-n)\gamma_1^2}{m(m-1)l - (m-1)(m\gamma_1^2 + \gamma_3^2)}\right| \tag{2.2}$$

$$l = \lambda^2 \gamma_1 I_3 g_1^{-1} = (-P \pm \sqrt{P^2 - 4Q})/2$$

$$P = [2mn - m - n + 1 + \gamma_1^2 (m-1)(m^2 - 2mn - m - n + 1)]\,[(m-1)mn]^{-1}$$

$$Q = (n-1)\gamma_3^2\,[1 - 3\gamma_1^2 (m-1)]\,[(m-1)mn]^{-1}$$

The discriminant $P^2 - 4Q$ is a total square in γ_3^2 only when $m = 2$ and $m + n = 1$. When $m = 2,\ 1 \leqslant n \leqslant 3$

$$l_1 = -\gamma_3^2, \quad l_2 = (n-1)(2 - 3\gamma_3^2)/2n$$

$$f_1 = |\gamma_3|, \quad f_2 = [n(3\gamma_3^2 - 2)/2(n-1)]^{1/2}$$

When $m + n = 1$ (i.e. in the case of a flate plate)

$$l_1 = \gamma_3^2/n, \quad l_2 = (1 + 3n\gamma_1^2)/n$$

$$f_1 = 2n\,|\gamma_3|\,/\,m, \quad f_2 = {}^1/_2\,(1 + 3n\gamma_1^2)^{1/2}$$

When $m + n \neq 1$, then in conformity with (2.2), as $\gamma_1 \to 0$

$$f_1 = 1 + {}^1/_2\,(m-1)\,[m(n-m) - 3(n-1)]\,(m+n-1)^{-1}\,\gamma_1^2 + O(\gamma_1^4)$$

$$f_2 = \left[\frac{n(m-1)}{m(n-1)}\right]^{1/2} + \left[\frac{m(m-1)(m-2)}{m+n-1} - {} + \frac{m^2 - m + 1}{2}\right]\gamma_1^2 + O(\gamma_1^4)$$

and when $\gamma_3 \to 0$ and $n \neq (m-1)^2/(2m-3)$

$$l_1 = \frac{n(2m-3) - (m-1)^2}{(m-1)\,n} + O(\gamma_3^2)$$

$$l_2 = \frac{(n-1)(3m-4)\gamma_3^2}{[(m-1)^2 - n(2m-3)]\,m} + O(\gamma_3^2)$$

$$f_1 = [(m-1)^2/n - (2m-3)]^{1/2}\,/\,(n - m + 1) + O(\gamma_3^2) \tag{2.3}$$

$$f_2 = (n - m + 1)\,[(3m-4)(1-m)]^{1/2}\,\{m(n-1)\,[(m-1)^2 - n(2m-3)]\}^{-1/2}\,|\gamma_3| + O(\gamma_3^2)$$

Functions $f_{1,2}(\gamma_3)$ are of a form similar to that in /4/, and are in the main determined by the values

$$f_{1,2}, \quad \frac{\partial f_{1,2}}{\partial \gamma_1^2} \quad \text{when} \quad \gamma_1 = 0 \quad \text{and}$$

$$f_{1,2}, \quad \frac{\partial f_1}{\partial \gamma_3^2}, \quad \frac{\partial f_2}{\partial \gamma_3} \quad \text{when} \quad \gamma_3 = 0$$

The signs plus and minus in Fig.1 indicate subregions of region (2.1) inside which $\left.\frac{\partial f_1}{\partial \gamma_1^2}\right|_{\gamma_1=0}$ retains positive and negative values, respectively. The curve in Fig.1, where this expression is zero, is defined by the equation $n = (m^2 - 3)/(m - 3)$, with the expansion of $f_1 - 1$ close to $\gamma_1 = 0$ beginning with terms of order γ_1^4. The similar curve for f_2 is defined by the equation

$$n = (m-1)(m^2 - 3m - 1)\,/\,(m^2 - m + 1)$$

Close to $\gamma_3 = 0$ the expansions of $f_{1,2}$ are defined by formulas (2.3). However on curve $n = (m-1)^2/(2m-3)$ the proportionality coefficient in the relation $f_2 \sim |\gamma_3|$ becomes infinite, and we then have

$$l = \text{const}\cdot|\gamma_3| + o(\gamma_3) \quad \text{and} \quad f_2 = \text{const}\cdot|\gamma_3|^{1/2} + o(|\gamma_3|^{1/2}).$$

3. Let us consider the general case in which the Staude curve K consists of four connected components adjoining the sphere poles where $\gamma_i = \pm 1$ and when it is possible to assume without loss of generality $g_1 > 0$ (123), $I_1 \neq I_2$ (123). These components are shown in Fig.2 by heavy lines on the unit sphere of directional cosines (the remaining notation is explained below).

At bifurcation points these components separate into sections with changed degree of instability /5/ of permanent rotations and those with unchanged number of sets of ellipses. Note also that on sections of curves adjacent to the poles corresponding to the semimean axis of the body ellipsoid of inertia, there exists one set, while on the semiminor and semimajor axes there are two.

Statement 3. As point (1.2) approaches along K the pole $\theta \to 0$ (or $\pm\pi$), one set of ellipses is transformed into a circle, and for the second, if it exists,

$$f_2 \to [I_{i+1} I_{i+2} (I_i - I_{i+1})^{-1} (I_i - I_{i+2})^{-1}]^{1/2}$$

Indeed, let us assume for definiteness $i = 1$, i.e. $\gamma_1 \to 1$, then retaining terms of the highest order in expressions for coefficients of the characteristic equation $\lambda^2(\lambda^4 + P\lambda^2 + Q) = 0$, we obtain

$$P = \omega^2 [1 + (I_1 - I_2)(I_1 - I_3) I_2^{-1} I_3^{-1}], \quad Q = \omega^4 (I_1 - I_2) \times (I_1 - I_3) I_2^{-1} I_3^{-1}$$

from which

$$\lambda_1^2 = -\omega^2 \leqslant 0, \quad \lambda_2^2 = -\omega^2 (I_1 - I_2)(I_1 - I_3) I_2^{-1} I_3^{-1}$$

If the first axis corresponds to the semimajor or semiminor axis of the ellipsoid of inertia, then $\lambda_2^2 \leqslant 0$. From (1.6) we have

$$\mathbf{R} = (-\lambda\omega d,\ 0,\ 0), \quad i\mathbf{S} = (0,\ g_1 d (I_3 - I_1)^{-1},\ g_3 d (I_1 - I_2)^{-1})$$
$$d = \alpha_1^* (\alpha_2^* \alpha_3^* - 1), \quad \alpha_1^* = \lim_{\gamma_1 \to 1} \alpha_1 < \infty \qquad (1\ 2\ 3)$$
$$\mathbf{z} = (0,\ g_3 (I_3 - I_1),\ g_1 (I_1 - I_2))$$

hence

$$\mathbf{u}' = \mathbf{u} = (0,\ \lambda\omega\gamma_3 d,\ -\lambda\omega\gamma_2 d) \parallel \mathbf{z}, \quad \mathbf{v} = \mathbf{v}' = (0,\ -\omega^2\gamma_2 d,\ -\omega^2\gamma_3 d) \perp \mathbf{z}$$
$$f = \frac{v}{u} = \frac{\omega}{|\lambda|}, \quad f_1 = 1, \quad f_2 = [I_2 I_3 (I_1 - I_2)^{-1} (I_1 - I_3)^{-1}]^{1/2}$$

Q.E.D.

Statement 4. As the point of the degree of instability change with zero characteristic root is approached, the ellipses contract to a segment directed along the tangent to K (i.e. $\theta \to \stackrel{+}{(-)} \pi/2,\ f \to 0$).

Indeed, when $\lambda = 0$ $\alpha_1 = 0$ (123), $\mathbf{S} = 0$, in conformity with (1.6)

$$\frac{1}{2}\mathbf{R} = (g_2\gamma_2\gamma_3^2 (I_1 - I_2)^{-1} + g_3\gamma_2^2\gamma_3 (I_3 - I_1)^{-1} - g_2\gamma_1\gamma_3^2 (I_1 - I_2)^{-1},\ \ -g_3\gamma_1\gamma_2^2 (I_3 - I_1)^{-1})$$

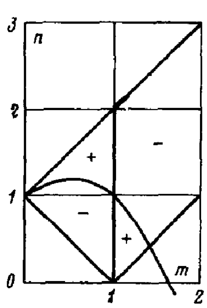

Fig.1

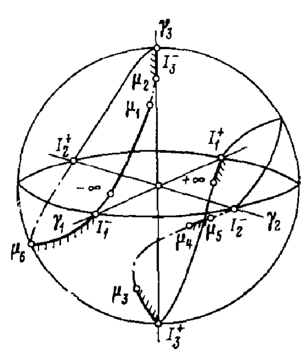

Fig.2

Then $\mathbf{R} \parallel \mathbf{z}$, $\mathbf{u} = [\mathbf{R}, \gamma] \perp \mathbf{z}$, $\mathbf{v} = 0$, and, consequently, $\theta = \stackrel{+}{(-)} \pi/2)$ $f = 0$. Another type of change of the degree of instability, when the discriminant $P^2 - 4Q$ changes its sign and $P < 0$, is also possible. Two sets of ellipses with common initial values of θ and f then appear, as of instance in the case of $I_1 = I_3$, $g_1/g_3 \leqslant 1$ /4/.

Statement 5. Functions θ and f become discontinuous at points where $u'^2 - v'^2$ vanishes. The increment of θ is then equal $\pi/2$ or $3\pi/2$, and f changes to $1/f$.

Proof. Let at the limit $u'^2 - v'^2 = 0$. Formula (1.7) implies that when $(\mathbf{u}', \mathbf{v}') \neq 0$, the limit values ahead and behind the discontinuity point are $\varphi_- = -\pi/4$, $\varphi_+ = \pi/4$, hence $\mathbf{u}_- = (\mathbf{u}' - \mathbf{v}')/\sqrt{2}$, $\mathbf{v}_- = (\mathbf{u}' + \mathbf{v}')/\sqrt{2}$, $\mathbf{u}_+ = \mathbf{v}_-$, $\mathbf{v}_+ = -\mathbf{u}_-$ and $(\mathbf{u}_-, \mathbf{u}_+) = (u'^2 - v'^2)/2 = 0$. The increment of angle θ is consequently $\Delta\theta = \pi/2$ or $3\pi/2$ (z changes continuously), and $f_- = |\mathbf{u}' + \mathbf{v}'|/|\mathbf{u}' - \mathbf{v}'| = 1/f_+$. When $(\mathbf{u}', \mathbf{v}') = 0$ then $\mathbf{u} = \mathbf{u}'$, $\mathbf{v} = \mathbf{v}'$, $u = v = 1$, $f = 1$. At the passage of f through the value $f = 1$ φ changes to $\varphi \pm \pi/2$ which leads to the change of $\mathbf{u}, \mathbf{v}$ to $\pm\mathbf{v}, \mp\mathbf{u}$ and to another jump of θ.

4. A complete analytical investigation of the behavior of functions θ and f on K presents considerable difficulties. These functions were investigated numerically for $I_1 = 6$, $I_2 = 4$, $I_3 = 3$, $g_1 = 1$, $g_2 = 0.01$, $g_3 = 0.02$. For a similar set of parameters the Staude curve has the largest sequence of sections with unchanged degree of instability.

Results of this investigation are shown in Figs.2 and 3. In Fig.2 the dash-dot line represents arcs of curve K, each of which constains one set of ellipses (and Liapunov periodic motions). Points on these arcs are of the saddle type for the reduced potential V, and unstable in the first approximation. Shading /in Fig.2/ denotes arcs with two sets of ellipses (and Liapunov periodic motion, when the nonresonance conditions $\lambda_1 \neq s\lambda_2$, where s is an integer, are satisfied). These are points of minimum or maximum of V . In case of minimum of V (shading lines directed downward) the last condition may be disregarded according to the theorem in /6/. Properties of critical points V on K are obtained in accordance with /3,7/.

The dependence of f and θ on μ which parametrizes curve K /8/

$$\gamma_1 = \frac{1}{\omega^2}\frac{g_1}{I_1 - \mu} \quad (1\,2\,3), \quad \omega^4 = \sum_{(1\,2\,3)} \frac{g_1^2}{(I_1 - \mu)^2}$$

is shown in Fig.3,a and b.

The solid and dash curves correspond to two different values of λ_1 and λ_2. The sign of θ is chosen so that $\theta > 0$ when the three vectors $\mathbf{u}, \mathbf{z}, \boldsymbol{\gamma}$ are right-handed. The symbols I_1+, I_1 , etc. in Fig.2 denote limit values of μ that correspond to the ends of admissible arcs and, also, to the points of change of the degree of instability $\mu_1, \ldots, \mu_6$, which are stationary for the integral of moment /5/

$$J(\mu) = \sum_{(1\,2\,3)} I_1\omega\gamma_1^2 = \omega^{-3}\sum_{(1\,2\,3)} \frac{I_1 g_1^2}{(I_1 - \mu)^2}$$

The dependence of J and λ_1/λ_2 on μ are shown in Fig.3,c by the continuous and dash lines, respectively.

The analysis of data shown in Fig.3 enables us to reach the following conclusions.

Functions θ and f are different for different frequencies at one and the same point.

Functions θ and f do not always vary monotonically and continuously. Their behavior conforms to Statements 3—5 and the remark. Hence, for instance $f \to \infty$ as $\mu \to \mu_1, \mu_4, \mu_5$, or μ_6.

The increment of θ on each arc is bounded by the quantity 3π and, briefly by $\pi/2$ on all arcs, except arc $(I_1, +\infty)$.

Functions f and λ_1/λ_2 are continuous when $\mu = I_i$.

Resonance points converge at points μ_2 and μ_3 (see Fig.3,c).

The degree of instability varies along K in dependence of the properties of V, and is denoted in Fig.3,c by the numerals (0), (1), and (2).

The author thanks V.V. Rumiantsev. V.G. Demin, and Ia.V., Tatarinov for useful remarks.

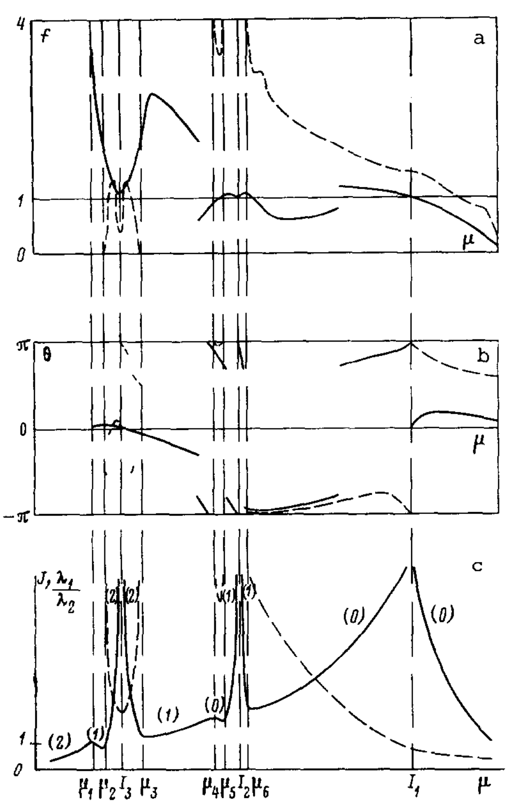

Fig.3

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Translated by J.J.D.